STA 331 2.0 Stochastic Processes

8. Birth and Death Processes

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Birth and Death Processes

- The birth-and-death process is a subclass of **continuous-time Markov chains**.
- The birth-and-death processes are characterized by the property that whenever a transition occurs from one state to another, then this transition can be to a neighbouring state only.

Transition types

- a transition occurs from one state to another and this transition can be to a neighbouring state only.
    - Eg: State space $S = \{0, 1, 2, ..., i, ...\}$
    - transition that occurs from state i, can be only to a neighboring state $(i - 1)$ or $(i + 1)$.

Birth rate and Death rate

Birth rate
λ_i - birth rate from state i, $i \geq 0$

Death rate
μ_i - death rate from state i, $i \geq 0$

Queueing systems

1. Birth - equivalent to the arrival of a customer.
2. Death - equivalent to the departure of a served customer.

Notations

A continuous-time Markov chain $[X(t), t \in T]$ with state space $S = \{0, 1, 2, ...\}$ with rates

$$q_{i,i+1} = \lambda_i, \;\; i = 0, 1, ...,$$

$$q_{i,i-1} = \mu_i, \;\; i = 1, 2, ...,$$

$$q_{i,j} = 0, \;\; j \neq i \pm 1, \;\; j \neq i, \;\; i = 0, 1, ..., \;\; \text{and}$$

$$q_i = (\lambda_i + \mu_i), \;\; i = 0, 1, ..., \;\; \text{and } \mu_0 = 0.$$

Pure birth process, pure death process, birth-and-death process

i) a pure birth process if $\mu_i = 0$ for $i = 1, 2, \ldots$
   - No decrements, only increments.

ii) a pure death process if $\lambda_i = 0$ for $i = 1, 2, \ldots$
   - No increments, only decrements.

iii) a birth-and-death process if some of the λ_i's and some of the μ_i's are positive.

Examples of random phenomena modelled through birth and death processes

- Spread of epidemic disease
- Mutant gene dynamics
- Cell kinetics (proliferation of cancer cells)

Special cases

1. Linear birth process: Yule-Furry process
2. Linear death process
3. Linear birth and death process
4. M/M/I queue

Pure Birth Process

- Special case of a **continuous-time Markov process** and a **generalisation of a Poisson process**.
- Consider a population of individuals where only the appearances of new individuals, which are called "birth" occur.

General birth processes

Let us consider a birth process whose total number of individuals at time t is denoted by a discrete random variable $N(t)$. As parameter t varies $\{N(t) : t \geq 0\}$ represents a stochastic process with a continuous parameter (time) space and a discrete state space.

Let us assume that the birth rate depends on the present size of the population. Further we assume that the births occur according to the following postulates:

$$P[N(t+h) = n+k|N(t) = n] = \begin{cases} \lambda_n h + o(h), & \text{k=1} \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k = 0 \end{cases}$$

General birth processes (cont)

Condition 1

$$P[N(t+h) = n+k|N(t) = n] = \begin{cases} \lambda_n h + o(h), & \text{k=1} \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k = 0 \end{cases}$$

where λ_n is the rate at which the births occur at time t and n being the size of the population at time t.

Condition 2

$N(0) > 0$

Your turn

Compare the differences in conditions between Poisson process, Non-homogeneous Poisson Process and Birth Process

Goal: Probability Mass Function of $N(t)$

What is the probability that the population size at a given time, t, equals $N(t) = n$?

$$P_n(t) = P[N(t) = n] =?$$

For example,

$$P_0(t) = P[N(t) = 0] =?$$

$$P_1(t) = P[N(t) = 1] =?$$

$$P_2(t) = P[N(t) = 2] =?$$

Linear Birth Process (Yule-Furry Process)

When, $\lambda_n = n\lambda$, i.e. when the birth rate is linear in the present size of the population.

Then the pure birth process is said to a **Linear Birth Process** or **Yule-Furry Process**.

Let is assume that **there is only one individual in the population initially,** $N(0) = 1$. It can be shown that for any $t > 0$.

$$P(N(t) = 0) = 0$$

$$P(N(t) = n) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \, , n \geq 1.$$

Proof (general situation):

For $n = 0$

$$P_0(t+h) = P(N(t)=0)P(N(t+h)=0|N(t)=0)$$

$$P_0(t+h) = P_0(t)(1-\lambda_0 h + o(h))$$

i.e.

$$P_0(t+h) = P_0(t) - \lambda_0 h P_0(t) + o(h)P_0(t)$$

$$lim_{h\to 0}\frac{P_0(t+h)-P_0(t)}{h} = -lim_{h\to 0}\lambda_0 P_0(t) + lim_{h\to 0}\frac{o(h)}{h}P_0(t)$$

i.e.

$$P_0'(t) = -\lambda_0 P_0(t).$$

We assume that **there is only one individual in the population initially,** $N(0) = 1$. Hence, $P[N(t) = 0] = 0$. That is $P_0(t) = 0$.

Proof: (cont)

For $n \geq 1$

$$\begin{aligned}
P_n(t+h) = &P(N(t)=n)P(N(t+h)=n|N(t)=n)+ \\
&P(N(t)=n-1)P(N(t+h)=n|N(t)=n-1)+ \\
&\sum_{r=2}^{n-1} P(N(t)=n-r)P(N(t+h)=n|N(t)=n-r)
\end{aligned}$$

i.e

$$\begin{aligned}
P_n(t+h) = &P_n(t)(1-\lambda_n h + o(h))+ \\
&P_{n-1}(t)(\lambda_{n-1}h + o(h))+ \\
&o(h)
\end{aligned}$$

Proof: (cont)

$P_n(t+h) = P_n(t) - \lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)$ for $n \geq 1$

$lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) + lim_{h \to 0} \frac{o(h)}{h}$

i.e.

$P'_n(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$ for $n \geq 1$.

Therefore the partial differential-difference equations is

For $n \geq 1$, $P'_n(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$.

When $n = 1$

$$P_1'(t) = -\lambda_1 P_1(t),$$
$$\int \frac{P_1'(t)}{P_1(t)} dt = -\lambda_1 \int dt,$$

$$lnP_1(t) = -\lambda_1 t + c$$
$$P_1(t) = c_1 e^{-\lambda_1 t}$$

When $t = 0$, $c_1 = 1$

$$P_1(t) = e^{-\lambda_1 t}$$

When $n = 2$

$$P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t),$$

$$P_2'(t) + \lambda_2 P_2(t) = \lambda_1 e^{-\lambda_1 t},$$

Multiply by $e^{\lambda_2 t}$

$$P_2'(t) e^{\lambda_2 t} + \lambda_2 P_2(t) e^{\lambda_2 t} = \lambda_1 e^{-\lambda_1 t} e^{\lambda_2 t},$$

$$\int \frac{d}{dt}[e^{\lambda_2 t} P_2(t)] dt = \int \lambda_1 e^{(\lambda_2 - \lambda_1)t} dt,$$

$$e^{\lambda_2 t} P_2(t) = \frac{\lambda_1 e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + c$$

When $t = 0$,

We know that $P_2(0) = 0$. hence,

$c = -\frac{\lambda_1}{\lambda_2 - \lambda_1}$.

Hence,

$$P_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} [e^{(\lambda_2 - \lambda_1)t} - 1]$$

Linear birth process (Yule-Furry Process)

When,

$$\lambda_n = n\lambda.$$

That is the birth rate is linear in the present size of the population.

Let us assume that there is **only one individual in the population** initially. That is $N(0) = 1$.

Then the difference-differential equations of the linear birth process takes the form

$P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$ for $n \geq 1$ with the initial conditions $P_1(0) = 1$ and $P_n(0) = 0$ for $n \geq 2$.

Linear birth process (Yule-Furry Process) (cont)

$P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$ for $n \geq 1$ with the initial conditions $P_1(0) = 1$ and $P_n(0) = 0$ for $n \geq 2$.

Multiplying the equation for n by z^n and summing over all n we obtain

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} P_n(t)z^n = -\lambda z \frac{\partial}{\partial z}\sum_{n=1}^{\infty} P_n(t)z^n + \lambda z^2 \frac{\partial}{\partial z}\sum_{n=1}^{\infty} P_{n-1}(t)z^{n-1}$$

Let $\prod(z,t) = \sum_{n=1}^{\infty} P_n(t)z^n$. Then the above equations becomes

$$\frac{\partial \prod(z,t)}{\partial t} = -\lambda z \frac{\partial \prod(z,t)}{\partial z} + \lambda z^2 \frac{\partial \prod(z,t)}{\partial z}$$

Linear birth process (Yule-Furry Process) (cont)

i.e. $\frac{\partial \prod(z,t)}{\partial t} = \lambda z(z-1)\frac{\partial \prod(z,t)}{\partial z}$

$\frac{\partial \prod(z,t)}{\partial t} - \lambda z(z-1)\frac{\partial \prod(z,t)}{\partial z} = 0$

Subsidiary equations take the form

$$\frac{dt}{1} = \frac{dz}{-\lambda z(z-1)} = \frac{d\prod}{0}$$

Two independent solutions can be obtained one from $d\prod = 0$ and the other from $-\lambda dt = \frac{dz}{z(z-1)}$.

$d\prod = 0 \Rightarrow \prod(z,t) = constant.$

$-\lambda dt = \frac{dz}{z(z-1)} \Rightarrow \frac{z}{z-1}e^{-\lambda t} = constant.$

Linear birth process (Yule-Furry Process) (cont)

The general solution can be written as

$\prod(z,t) = f\left(\frac{z}{z-1}e^{-\lambda t}\right)$ where f is an arbitrary function.

The initial conditions $P_1(0) = 1$ and $P_n(0) = 0$ for $n \geq 2$ imply that $\prod(z,0) = z$.

$$\therefore \prod(z,0) = f\left(\frac{z}{z-1}\right) = z.$$

Let $\omega = \frac{z}{z-1} \Rightarrow z = \frac{\omega}{\omega-1}$ and hence we obtain $f(\omega) = \frac{\omega}{\omega-1}$.

Linear birth process (Yule-Furry Process) (cont)

$$\therefore \prod(z,t) = \frac{\frac{z}{z-1}e^{-\lambda t}}{\frac{z}{z-1}e^{-\lambda t} - 1} = \frac{ze^{-\lambda t}}{ze^{-\lambda t} - (z-1)} = \left(1 - \frac{z-1}{z}e^{-\lambda t}\right)^{-1}$$

Considering coefficients of z^n we have

$$P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \text{ for } n \geq 1.$$

In proving the above results we assume that initially there is only one individual in the population. That is N(0)=1.

Now let's prove for the case $N(0) = a, a \geq 1$. For that we use moment generating functions.

Moment generating function of $N(t)$

Let

$$M_{N(t)}(\theta, t) = E[e^{N(t)\theta}],$$

be the moment generating function of $N(t)$. Then, for $t > 0$,

$$\begin{aligned} M_{N(t)}(\theta, t) &= \sum_{n=0}^{\infty} e^{n\theta} P(N(t) = n) \\ &= \sum_{n=0}^{\infty} e^{n\theta} P_n(t). \end{aligned} \tag{1}$$

Moment generating function of $N(t)$ (cont.)

We assume that $N(0) = a > 0$. Hence, $P_n(t) = 0$ for all $n < a$. Hence,

$$M_{N(t)}(\theta, t) = \sum_{n=a}^{\infty} e^{n\theta} P_n(t). \tag{2}$$

Moment generating function of $N(t)$ (cont.)

Now we take derivative w.r.t θ. Then we get,

$$\frac{\partial}{\partial\theta} M_{N(t)}(\theta, t) = \sum_{n=a}^{\infty} n e^{n\theta} P_n(t).$$

The derivative w.r.t t is

$$\begin{aligned}
\frac{\partial}{\partial t} M_{N(t)}(\theta, t) &= \sum_{n=a}^{\infty} e^{n\theta} P_n'(t) \\
&= \sum_{n=a}^{\infty} e^{n\theta} [-n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)] \\
&= -\sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{n=a}^{\infty} (n-1) e^{n\theta} \lambda P_{n-1}(t)
\end{aligned} \tag{3}$$

Moment generating function of $N(t)$ (cont.)

Since $P_{a-1}(t) = 0$, the second summation starts at $a+1$. Hence,

$$\begin{aligned}
\frac{\partial}{\partial t} M_{N(t)}(\theta, t) &= -\sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{n=a+1}^{\infty} (n-1) e^{n\theta} \lambda P_{n-1}(t) \\
&= -\sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{m=a}^{\infty} m e^{(m+1)\theta} \lambda P_m(t) \\
&= -\lambda \sum_{n=a}^{\infty} n e^{n\theta} P_n(t) + \lambda e^{\theta} \sum_{m=a}^{\infty} m e^{m\theta} P_m(t) \\
&= -\lambda \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) + \lambda e^{\theta} \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) \\
&= \lambda (e^{\theta} - 1) \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t)
\end{aligned} \tag{4}$$

Moment generating function of $N(t)$ (cont.)

$$\frac{\partial}{\partial t} M_{N(t)}(\theta, t) - \lambda(e^{\theta} - 1)\frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = 0. \tag{5}$$

Note:
A partial differential equation (PDE) for a function $z(x, y)$ is Lagrange type if it takes the form (General form of first-order quasilinear PDE)

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z). \tag{6}$$

The associated characteristic system of ordinary differential equations.

Note (cont)

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}. \tag{7}$$

is known as the characteristic (auxiliary) system of equation (5). Suppose that two independent particular solutions of this system have been found in the form

$u(x, y, z) = C_1$ and $v(x, y, z) = C_2$, where where C_1 and C_2 are arbitrary constants.

Then the general solution to equation (5) can be written as

$$\phi(u, v) = 0 \tag{8}$$

where ϕ is an arbitrary function of two variables.

Note (cont.)
With equation (6) solved for v, one often specifies the general solution in the form $v = \psi(u)$, where $\psi(u)$ is an arbitrary function of one variable. The ψ can be determined using the boundary conditions.

Moment generating function of $N(t)$ (cont.)

Revisit equation 4,

$$\frac{\partial}{\partial\theta} M_{N(t)}(\theta, t) - \lambda(e^{\theta} - 1)\frac{\partial}{\partial\theta} M_{N(t)}(\theta, t) = 0. \tag{9}$$

According to the auxiliary system of equation in (6),

$$\frac{dt}{1} = \frac{d\theta}{-\lambda(e^{\theta} - 1)} = \frac{M_{N(t)}}{0}$$

$$\frac{dt}{1} = \frac{dM_{N(t)}}{0}$$

$\frac{dM_{N(t)}}{dt} = 0 \Rightarrow M_{N(t)}(\theta, t) = constant.$

Moment generating function of $N(t)$ (cont.)

Furthermore consider,

$$\frac{dt}{1} = \frac{d\theta}{-\lambda(e^{\theta} - 1)}$$

$$\begin{aligned} \lambda dt &= -\frac{1}{(e^{\theta} - 1)} d\theta \\ &= \frac{-e^{-\theta}}{1 - e^{-\theta}} d\theta \end{aligned} \tag{10}$$

From equation (9) we can write

$$\lambda t = -ln(1 - e^{-\theta}) + c$$

,

Moment generating function of $N(t)$ (cont.)

Furthermore

$$ln(e^{\lambda t}) + ln(1 - e^{-\theta}) = c.$$

Hence,

$$e^{\lambda t}(1 - e^{-\theta}) = constant.$$

Hence, the general solution for eq(8) is

$$M_{N(t)}(\theta, t) = \Psi[e^{\lambda t}(1 - e^{-\theta})].$$

Moment generating function of $N(t)$ (cont.)

The boundary conditions $P_a(0) = 1$, and $P_n(0)$ for $n \neq a$, imply that $M_{N(t)}(\theta, 0) = \sum_{n=a}^{\infty} e^{n\theta} P_n(0) = e^{a\theta}$,

$$M_{N(t)}(\theta, 0) = e^{a\theta} = \Psi(1 - e^{-\theta}).$$

Let $\alpha = 1 - e^{-\theta}$. Then, $e^{\theta} = (1 - \alpha)^{-1}$. Hence,

$$e^{a\theta} = \Psi(\alpha) = (1 - \alpha)^{-a}.$$

Moment generating function of $N(t)$ (cont.)

Therefore,

$$M_{N(t)}(\theta, t) = \Psi[e^{\lambda t}(1 - e^{-\theta})] = [1 - e^{\lambda t}(1 - e^{-\theta})]^{-a}.$$

Let $p = e^{-\lambda t}$ and $p + q = 1$. Then,

$$M_{N(t)}(\theta, t) = [1 - p^{-1}(1 - e^{-\theta})]^{-a} = \left[\frac{p - 1 + e^{-\theta}}{p}\right]^{-a} = \left(\frac{p}{e^{-\theta} - q}\right)^{a}$$

Now from this MGF, we can derive the moments of $N(t)$.

Moment generating function of $N(t)$ (cont.)

It can be shown that

$E(N(t)) = a/p = ae^{\lambda t}$ and

$V[N(t)] = a(1-p)/p^2 = a(1-e^{-\lambda t})e^{2\lambda t}$.

Furthermore, we recognize the above MGF is in the form of the MGF of a negative binomial random variable Y, with probability mass function
$P(Y = y) =^{y-1} C_{a-1} p^{a-1} q^{y-1-(a-1)} p =^{y-1} C_{a-1} p^a q^{y-a}, \quad$ for $y = a, a+1, \ldots$

Hence,

$P(N(t) = n) =^{n-1} C_{a-1} p^a q^{n-a} =^{n-1} C_{a-1} e^{-\lambda t a}(1-e^{-\lambda t})^{n-a}$ for $n = a, a+1, \ldots$

Linear birth process (Yule-Furry Process)

Summary:

When, $N(0) = 1$

$P(N(t) = 0) = 0$

$P(N(t) = n) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$, $n \geq 1$.

When, $N(0) = a$

$P(N(t) = n) =^{n-1} C_{a-1} p^a q^{n-a} =^{n-1} C_{a-1} e^{-\lambda ta}(1 - e^{-\lambda t})^{n-a}$ for $n = a, a+1, ...$

Exercise

Consider a pure birth process on the states $\{0, 1, ..., N\}$ for which $\lambda_k = (N - k)\lambda$ for $k = 0, 1, ..., N$. Suppose $N(0) = 0$. Find $Pn(t) = P(X(t) = n)$ for $n = 0, 1$ and 2.