

# STA 331 2.0 Stochastic Processes

## 3. Markov Chains - Classification of States

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# Example 1

Find the equivalence classes.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Theorem

The relation of communication partitions the state space into mutually exclusive and exhaustive classes. (The states in a given class communicate with each other. But states in different classes do not communicate with each other.)

# Definition

Let  $f_i$  denote the probability that, starting in state  $i$ , the process will ever re-enters state  $i$ , i.e.,

$$f_i = P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$$

## Example 2

Consider the Markov chain consisting of the states 0, 1, 2, 3 with the transition probability matrix,

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find  $f_0, f_1, f_2, f_3$ .

## Recurrent and transient states

Let  $f_i$  be the probability that, starting in state  $i$ , the process will ever re-enter state  $i$ . State  $i$  is said to be recurrent if  $f_i = 1$  and transient if  $f_i < 1$ .

## Example 3

Consider the Markov chain consisting of the states 0,1,2 with the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Determine which states are transient and which are recurrent.

## Example 4

Consider the Markov chain consisting of the states 0, 1, 2, 3 with the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Determine which states are transient and which are recurrent.



## Example 5

Consider the Markov chain consisting of the states 0, 1, 2, 3, 4 with the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Determine which states are transient and which are recurrent.

# Theorem

if state  $i$  is recurrent then, starting in state  $i$ , the process will re-enter state  $i$  again and again and again—in fact, infinitely often.

# Theorem

For any state  $i$ , let  $f_i$  denote the probability that, starting in state  $i$ , the process will ever re-enter state  $i$ . If state  $i$  is transient then, starting in state  $i$ , the number of time periods that the process will be in state  $i$  has a geometric distribution with finite mean  $\frac{1}{1-f_i}$ .

Proof: In-class



# Theorem

State  $i$  is

recurrent if  $\sum_{n=1}^{\infty} P_{ii}^n = \infty,$

transient if  $\sum_{n=1}^{\infty} P_{ii}^n < \infty,$

Proof: In-class

## Corollary 1

If state  $i$  is recurrent, and state  $i$  communicates with state  $j$  ( $i \leftrightarrow j$ ), then state  $j$  is recurrent.

Proof: In-class

## Corollary 2

In a Markov Chain with a finite number of states not all of the states can be transient (There should be at least one recurrent state).

Proof: In-class

## Corollary 3

If one state in an equivalent class is transient, then all other states in that class are also transient.

Proof: In-class



## Corollary 4

Not all states in a finite Markov chain can be transient. This leads to the conclusion that **all states of a finite irreducible Markov chain are recurrent.**

Proof: In-class

# Acknowledgement

The contents in the slides are mainly based on Introduction to Probability Models by Sheldon M. Ross.