

# STA 331 2.0 Stochastic Processes

## 11. Birth-and-Death Process - important results (cont)

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# Definition

A continuous parameter stationary Markov process is a stochastic process having the properties that

1. Each time it enters state  $i$ , the amount of time it spends in that state before making a transition into a different state is exponentially distributed (say with rate  $\nu_i$  or mean  $\frac{1}{\nu_i}$ ), and
2. When the process leaves state  $i$ , it enters state  $j$  with some probability,  $p_{ij}$  satisfying,

$$P_{ii} = 0 \text{ all } i$$
$$\sum_j P_{ij} = 1 \text{ all } i$$

## Birth-and-death process

For birth and death process, let  $\lambda_i$  and  $\mu_i$  be given by

$$\lambda_i = q_{i,i+1} \text{ and } \mu_i = q_{i,i-1}.$$

The values  $\{\lambda_i, i \geq 0\}$  and  $\{\mu_i, i \geq 0\}$  are called respectively the birth and death rate. Then

$$\nu_i = \lambda_i + \mu_i$$

Then  $T_i \sim \exp(\lambda_i + \mu_i)$ .

Furthermore,

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}$$

## Examples of birth-and-death process

Example 1:

Suppose that life-time of a component of a machine is exponentially distributed with rate  $\lambda$ .

Let  $X(t)$  be the state of the machine at time  $t$ .

$$X(t) = \begin{cases} 1, & \text{if the machine is operational at time } t \\ 0, & \text{if the machine is not operational at time } t \end{cases} \quad (1)$$

This is a continuous parameter discrete state Markov process with absorbing barrier state 0 (suppose that there are no repairs).

# Examples of birth-and-death process

Example 1 (cont.):

**Instantaneous transition probabilities**

$$P_{01} = 0$$

and

$$P_{10} = 1$$

.

Hence,  $P_{11} + P_{10} = 1$ .

# Examples of birth-and-death process

Example 1 (cont.):

Your turn: calculate the associate probabilities for,

$$P_1(t) = P(X(t) = 1) = ?$$

$$P_0(t) = P(X(t) = 0) = ?$$

## Examples of birth-and-death process

Example 1 (cont.):

**second type of transition probability  $P_{ij}(t)$ .**

$$P_{10}(t) = P(X(t+s) = 0 | X(s) = 1).$$

Write this in terms of  $T_1$

$$P_{10}(t) = P(T_1 \leq t).$$

$$T_1 \sim \text{exp}(\lambda)$$

You can show that  $P_{10}(t) = 1 - e^{-\lambda t}$  and

$$P_{11}(t) = e^{-\lambda t}.$$

## Examples of birth-and-death process

Example 2:

A machine is operational for a time that is exponentially distributed with rate  $\alpha$  and off or down for a time that is exponentially distributed with rate  $\beta$ . For example, the machine needs a part that has an exponentially distributed lifetime; once it burns out, the fix-it time (time required to obtain a new part) is also exponentially distributed.

$$X(t) = \begin{cases} 1, & \text{if the machine is operational at time } t \\ 0, & \text{if the machine is not operational at time } t \end{cases} \quad (2)$$

This is a continuous parameter discrete state Markov process  
There is no absorbing barrier state.



# Examples of birth-and-death process

Example 2:

Instantaneous transition prob.	PMF	Transition Prob.
$P_{01}$	$P_1(t)$	$P_{00}(t)$ $P_{01}(t)$
$P_{01}$	$P_0(t)$	$P_{10}(t)$ $P_{11}(t)$

$T_1 \sim \exp(\alpha)$  and  $T_0 \sim \exp(\beta)$ .

$T_0$  - time it takes to repair the component.

## Examples of birth-and-death process

### Example 3 (One server queue)

Suppose that there is **one** checkout counter at a shopping store. Suppose that customers arrive to a single server system according to a Poisson processing having rate  $\lambda$  and the service time is exponentially distributed with parameter  $\nu$ .

## Examples of birth-and-death process

### Example 4 (Two server queue)

Suppose that there are **two** parallel identical checkout counter at a shopping store. The service times are independently and identically distributed. Suppose that customers arrive according to a Poisson process having rate  $\lambda$  and the service time is exponentially distributed with parameter  $\nu$ .

## Examples of birth-and-death process

### Example 5

A telephone operator has a phone with a hold button. Suppose incoming voice calls arrive according to a Poisson process with rate  $\lambda$ . Each call takes an exponentially distributed time with average  $1/\nu$  minutes. If a call arrives during a time the phone is busy, it is placed on hold. If another call arrives, it receives a busy tone and must hang up.