STA 331 2.0 Stochastic Processes

11. Birth-and-Death Process

Dr Thiyanga S. Talagala

November 24, 2020

Department of Statistics, University of Sri Jayewardenepura

Birth-and-death process

Consider the stochastic process with $\{N(t), t \geq 0\}$ with $N(0) = a(\geq 1)$, and

$$P[N(t+h) = n+k | N(t) = n] = \begin{cases} 1 - \lambda_n h - \mu_n h + o(h), & k = 0 \\ \lambda_n h + o(h), & \text{k=1} \\ \mu_n h + o(h), & \text{k=-1} \\ o(h), & k \geq 2 \text{ or } k \leq -2 \end{cases} \quad (1)$$

is called a birth and death process. Note $\mu_0 = 0$

Birth-and-death process

The partial differential-difference equations are

$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$ and

$P_n'(t) = -(\mu_n + \lambda_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$ for $n \geq 1$.

Proof

Let $P_n(t) = P[N(t) = n]$

Then, for $n \geq 1$,

$$\begin{aligned}
P_n(t+h) = {} & P(N(t) = n)P(N(t+h) = n | N(t) = n) + \\
& P(N(t) = n+1)P(N(t+h) = n | N(t) = n+1) + \\
& P(N(t) = n-1)P(N(t+h) = n | N(t) = n-1) + \\
& \sum_{r \neq -1,0,1}^{\infty} P(N(t) = n-r)P(N(t+h) = n | N(t) = n-r)
\end{aligned}$$

Proof (cont.)

$$\begin{aligned}P_n(t+h) = {} & P_n(t)(1-\mu_n h-\lambda_n h+o(h))+\\ & P_{n+1}(t)(\mu_{n+1}h+o(h))+\\ & P_{n-1}(t)(\lambda_{n-1}h+o(h))+\\ & o(h).\end{aligned}$$

$$\begin{aligned}P_n(t+h) = {} & P_n(t)-\mu_n P_n(t)h-\lambda_n P_n(t)h+\\ & P_{n+1}(t)\mu_{n+1}h+P_{n-1}(t)\lambda_{n-1}h+\\ & o(h) \text{ for } n\geq 1.\end{aligned}$$

Proof (cont.)

$$lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h} = -\mu_n P_n(t) - \lambda_n P_n(t) + P_{n+1}(t)\mu_{n+1} + P_{n-1}(t)\lambda_{n-1} + lim_{h\to 0}\frac{o(h)}{h} \text{ for } n \geq 1.$$

$$lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h} = -(\mu_n + \lambda_n) P_n(t) + P_{n+1}(t)\mu_{n+1} + P_{n-1}(t)\lambda_{n-1} + lim_{h\to 0}\frac{o(h)}{h} \text{ for } n \geq 1.$$

$P_n'(t) = -(\mu_n + \lambda_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$ for $n \geq 1$.

Proof (cont.)

For $n = 0$

$$
\begin{aligned}
P_0(t+h) = P(N(t) = 0)P(N(t+h) = 0|N(t) = 0) + \\
P(N(t) = 1)P(N(t+h) = 0|N(t) = 1) + \\
\sum_{r=2}^{\infty} P(N(t) = n-r)P(N(t+h) = 0|N(t) = n-r)
\end{aligned}
$$

$$
\begin{aligned}
P_0(t+h) = P_0(t)(1 - \mu_0 h - \lambda_0 h + o(h)) + \\
P_1(t)(\mu_1 h + o(h)) + \\
o(h)
\end{aligned}
$$

Proof (cont.)

We know that $\mu_0 = 0$

$$\begin{aligned} P_0(t+h) = P_0(t)(1 - \lambda_0 h + o(h)) + \\ P_1(t)(\mu_1 h + o(h)) + \\ o(h) \end{aligned}$$

$$\begin{aligned} P_0(t+h) = P_0(t) - P_0(t)\lambda_0 h + P_0(t)o(h) + \\ P_1(t)\mu_1 h + P_1(t)o(h) + \\ o(h) \end{aligned}$$

Proof (cont.)

$$lim_{h\to 0}\frac{P_0(t+h)-P_0(t)}{h} = -\lambda_0 P_0(t) + P_1(t)\mu_1 + lim_{h\to 0}\frac{o(h)}{h} \text{ for } n \geq 1.$$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

Linear Birth and Death Process

When $\lambda_n = n\lambda$ and $\mu_n = n\mu$, i.e when the birth and death rates are linear in the present size of the population, the birth and death process is said to be a linear birth and death process. Let us assume $N(0) = a(\geq 1)$.

Birth and death process takes the form

$P_0'(t) = \mu P_1(t)$

$P_n'(t) = -n(\mu + \lambda)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$ for $n \geq 1$.

Q1

We are going to show that

$$E[N(t)] = ae^{\lambda-\mu}t.$$

Q1 (proof)

$$P_0'(t) = \mu P_1(t)$$

$$P_n'(t) = -n(\mu+\lambda)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

for $n \geq 1$.

Using the same method as in pure birth process, we can show that

$$\frac{\partial}{\partial t} M_{N(t)}(\theta, t) - [\lambda(e^{\theta} - 1) + \mu(e^{-\theta} - 1)] \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = 0 \tag{2}$$

Q1 (proof)

The auxiliary system is

$$\frac{dt}{1} = \frac{-d\theta}{[\lambda(e^{\theta} - 1) + \mu(e^{-\theta} - 1)]} = \frac{dM_{N(t)}}{0},$$

$$\frac{dM_{N(t)}}{0} = 0$$

$$\Rightarrow M_{N(t)}(\theta, t) = constant.$$

Q1 (proof):

$$dt = \frac{-d\theta}{[\lambda(e^{\theta} - 1) + \mu(e^{-\theta} - 1)]}$$

$$t = \begin{cases} -\frac{1}{\lambda-\mu} ln \frac{(e^{-\theta}-1)}{\lambda e^{\theta}-\mu} + constant, & \text{when } \lambda \neq \mu \\ -\frac{1}{\lambda(e^{\theta}-1)} + constant, & \text{when } \lambda = \mu \end{cases} \tag{3}$$

$\Rightarrow$

$$\frac{(e^{\theta} - 1)e^{(\lambda-\mu)t}}{\lambda e^{\theta} - \mu} = constant \text{ when } \lambda \neq \mu$$

$$\lambda t - \frac{1}{(e^{\theta} - 1)} = constant \text{ when } \lambda = \mu.$$

Q1 Proof(cont.)

When $\lambda \neq \mu$, the general solution of eq 2 is

$$M_{N(t)}(\theta, t) = \Psi\left(\frac{(e^{\theta}-1)e^{(\lambda-\mu)t}}{\lambda e^{\theta}-\mu}\right). \qquad (4)$$

Initially there are, a, individuals ($N(0) = a$). Hence boundary conditions are $P_a(0) = 1$ and $P_n(0) = 0$ for $n \neq a$. Hence,

$$M_{N(t)}(\theta, 0) = \sum_{n=-\infty}^{\infty} e^{n\theta} P_n(0) = e^{a\theta}.$$

Therefore,

$$M_{N(t)}(\theta, 0) = e^{a\theta} = \Psi[\frac{e^{\theta}-1}{\lambda e^{\theta}-\mu}]. \qquad (5)$$

Q1 (proof)

Let,

$$\alpha = \frac{e^\theta - 1}{\lambda e^\theta - \mu}.$$

Then we get

$$e^\theta = \frac{\mu\alpha - 1}{\lambda\alpha - 1}.$$

Q1 (proof)

Substitute to eq 5, we get

$$\Psi(\alpha) = \left(\frac{\mu\alpha - 1}{\lambda\alpha - 1}\right)^a. \tag{6}$$

From eq 4 we have,

$$M_{N(t)}(\theta, t) = \Psi\left(\frac{(e^\theta - 1)(e^{(\lambda-\mu)t})}{\lambda e^\theta - \mu}\right).$$

Let $\nu(\theta, t) = \frac{(e^\theta - 1)e^{(\lambda-\mu)t}}{\lambda e^\theta - \mu}$. Therefore,

$$M_{N(t)}(\theta, t) = \left(\frac{\mu\nu(\theta, t) - 1}{\lambda\nu(\theta, t) - 1}\right)^a. \text{ (similar to eq 6 format)}$$

Q1 (proof): Your turn

Using the MGF, show that

$$E(N(t)) = ae^{(\lambda-\mu)t}.$$